LESSON 2 Negative exponents • Product and power theorems for exponents • Circle relationships

2.A negative exponents

Negative exponents cannot be "understood" because they are the result of a definition, and thus there is nothing to understand. We define 2 to the third power as follows:

$$2^3 = 2 \cdot 2 \cdot 2$$

We have agreed that 2^3 means 2 times 2 times 2. In a similar fashion, we define 2 to the negative third power to mean 1 over 2 to the third power.

$$2^{-3} = \frac{1}{2^3}$$

Thus, we have two ways to write the same thing. We give the formal definition of negative exponents as follows:

> DEFINITION OF x^{-n}
>
> If n is any real number and x is any real number that is not zero,
>
> $$x^{-n} = \frac{1}{x^n}$$

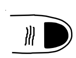
This definition tells us that when we write an exponential expression in reciprocal form, the sign of the exponent must be changed. If the exponent is negative, it is positive in reciprocal form; and if it is positive, it is negative in reciprocal form. In the definition we say that x cannot be zero because division by zero is undefined.

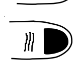
example 2.1 Simplify: (a) $\frac{1}{3^{-2}}$ (b) 3^{-3} (c) -3^{-2} (d) $(-3)^{-2}$ (e) $-(-3)^{-3}$

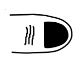
solution (a) $\frac{1}{3^{-2}} = 3^2 = \mathbf{9}$ (b) $3^{-3} = \frac{1}{3^3} = \mathbf{\frac{1}{27}}$

(c) Negative signs and negative exponents in the same expression can lead to confusion. If the negative sign is not "protected" by parentheses, a good ploy is to cover the negative sign with a finger. Then simplify the resulting expression and remove the finger as the last step.

-3^{-2} problem

 3^{-2} covered minus sign

 $\frac{1}{3^2}$ equivalent expression

 $\frac{1}{9}$ simplified

$-\mathbf{\frac{1}{9}}$ removed finger

(d) When we try to slide our finger over the minus sign in (d), we find that we cannot because the minus sign is "protected" by the parentheses.

$(-3)^{-2}$ problem

$(-3)^{-2}$	"protected"
$\dfrac{1}{(-3)^2}$	equivalent expression
$\mathbf{\dfrac{1}{9}}$	simplified

(e) One of the minus signs is "unprotected."

$-(-3)^{-3}$	problem
$(-3)^{-3}$	covered minus sign
$\dfrac{1}{(-3)^3}$	equivalent expression
$-\dfrac{1}{27}$	simplified
$-\left(-\dfrac{1}{27}\right) = \mathbf{\dfrac{1}{27}}$	removed finger

2.B product theorem for exponents

We remember that x^2 means x times x

$$x^2 = x \cdot x$$

and x^3 means x times x times x

$$x^3 = x \lozenge x \lozenge x$$

Using these definitions, we can find an expression whose value equals to x^2 times x^3.

$x^2 \cdot x^3$ means $x \cdot x$ times $x \cdot x \cdot x$ which equals x^5

This demonstrates the product theorem for exponents, which we state formally in the following box.

> **Product Theorem for Exponents**
>
> If m and n and x are real numbers and $x \neq 0$,
>
> $$x^m \cdot x^n = x^{m+n}$$

This theorem holds for all real number exponents.

example 2.2 Simplify: $x^2yx^{-5}y^{-4}x^5x^0$

solution We simplify by adding the exponents of like bases and get

$$\mathbf{x^2y^{-3}}$$

example 2.3 Simplify: $\dfrac{yy^{-3}x^4y^5x^{-10}}{y^{-6}x^{-3}y^{10}x^2}$

solution First we simplify the numerator and the denominator. Then we decide to write the answer with all factors in the numerator.

$$\frac{y^3x^{-6}}{y^4x^{-1}} = \mathbf{y^{-1}x^{-5}}$$

2.C power theorem for exponents

We can use the product theorem to expand $(x^2)^3$ as

$$(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$$

This procedure generalizes to the power theorem for exponents.

> **Power Theorem for Exponents**
>
> If m and n and x are real numbers,
>
> $$(x^m)^n = x^{mn}$$

This theorem can be extended to any number of exponential factors.

> **Extension of the Power Theorem**
>
> If the variables are real numbers,
>
> $$(x^m y^a z^b k^c \ldots)^n = x^{mn} y^{an} z^{bn} k^{cn} \ldots$$

example 2.4 Simplify: $\dfrac{x(x^{-3})^2 y(xy^{-2})^{-3}}{(y^2)^3 y^{-3}(x^2)^3}$

solution First we will use the power theorem in both the numerator and the denominator and get

$$\frac{xx^{-6}yx^{-3}y^6}{y^6y^{-3}x^6}$$

Now we simplify both the numerator and the denominator, and as the last step, we decide to write all exponential expressions with positive exponents.

$$\frac{x^{-8}y^7}{y^3x^6} = \boldsymbol{\frac{y^4}{x^{14}}}$$

2.D circle relationships

If we know the area of a circle, we can find the diameter of the circle and can find the radius of the circle. If we know the circumference of a circle, we can also find the diameter and the radius of the circle.

example 2.5 The area of a circle is 12.2 m^2. What is the approximate circumference of the circle?

solution First we find the radius.

$\pi r^2 = \text{area}$	equation
$\pi r^2 = 12.2$	substituted
$r^2 = \dfrac{12.2}{\pi}$	divided by π
$r = \sqrt{\dfrac{12.2}{\pi}}$	square root of both sides
$r \approx 1.97 \text{ m}$	simplified

We used a calculator and rounded the answer to two decimal places, so the answer is not exact. We indicate that the answer is not exact by using the symbol ≈ for "approximately equal to." The circumference equals $2\pi r$, so now we can find the circumference.

$$\begin{aligned} \text{Circumference} &= 2\pi r && \text{equation} \\ &\approx 2\pi(1.97) && \text{substituted} \\ &\approx \mathbf{12.38\ m} && \text{simplified} \end{aligned}$$

example 2.6 The circumference of a circle is 8π cm. What is the area of the circle?

solution First we find the radius.

$$\begin{aligned} \text{Circumference} &= 2\pi r && \text{equation} \\ 8\pi &= 2\pi r && \text{substituted} \\ \frac{8\pi}{2\pi} &= r && \text{divided by } 2\pi \\ 4\text{ cm} &= r && \text{simplified} \end{aligned}$$

Now we can use 4 cm for r to find the area.

$$\begin{aligned} \text{Area} &= \pi r^2 && \text{equation} \\ &= \pi(4\text{ cm})^2 && \text{substituted} \\ &= \mathbf{16\pi\ cm^2} && \text{simplified} \end{aligned}$$

practice

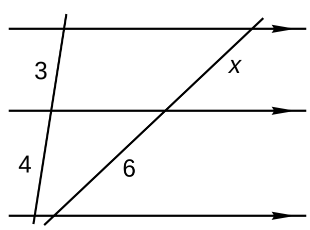
Simplify:

a. -4^{-2} **b.** $-(-4)^{-2}$ **c.** $\dfrac{(x^2y^{-2})^0(x^{-3}y)^{-2}}{y^{-8}x^4y^2x^3}$

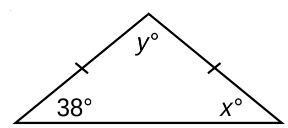
d. The area of a circle is 49π cm^2. What is the circumference of the circle?

problem set 2

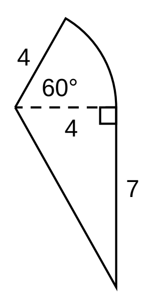
1. Find x.



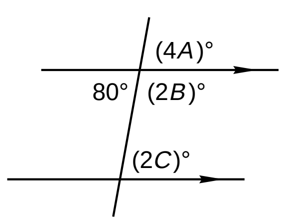
2. Find x and y.



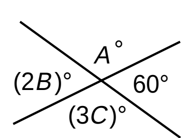
3. The base of a cylinder is a right triangle topped by a 60° sector of a circle, as shown. If the dimensions are in meters and the height of the cylinder is 8 meters, what is the volume of the cylinder?



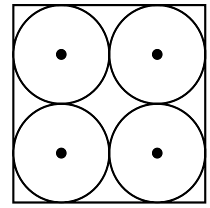
4. Find A, B, and C.



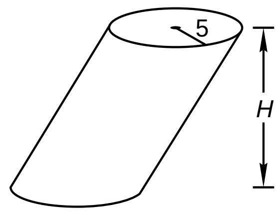
5. Find A, B, and C.



6. The area of the square is 16 cm^2. What is the length of one side? The circles inside the square are all the same size. What is a radius of one circle? What is the area of one circle?



7. The volume of this circular cylinder
c o
is 250π cm^3. What is the height of the cylinder? Dimensions are in centimeters.



8. The figure shown is the base of a
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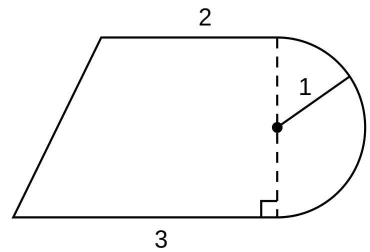
whose altitude is 4 meters. What is the volume of the cone? Dimensions are in meters.



Simplify. Write answers with all exponential expressions in the numerator.

9. $\dfrac{xx^2(x^0y^{-1})^2}{x^2x^{-5}(y^2)^5}$

10. $\dfrac{m^2p^0(m^{-2}p)^2}{m^{-2}p^{-1}(m^{-3}p^2)^3}$

11. $\dfrac{(x^2y)^0xy}{x^2(y^{-2})^3}$

12. $\dfrac{(a^2b^0)^2ab^{-2}}{a^2b^{-2}(ab^{-3})^2}$

Simplify. Write answers with positive exponents.

13. $\dfrac{(xm^{-1})^{-3}x^2m^2}{(x^0y^2)^{-2}xy}$

14. $\dfrac{(c^2d)^{-3}c^{-5}}{(c^2d^0)^{-2}d^3}$

15. $\dfrac{(m^2n^{-5})^{-2}m(n^0)^2}{(m^2n^{-2})^{-3}m^2}$

16. $\dfrac{(x^{-2}y^5)^3(x^2)^0y}{xy^{-3}x^{-2}}$

17. $\dfrac{(b^2c^{-2})^{-3}c^{-3}}{(b^2c^0b^{-2})^4}$

Simplify. Write answers with negative exponents.

18. $\dfrac{(abc)^{-3}c^2b}{a^{-4}bc^2a}$

19. $\dfrac{kL^2k^{-2}}{(k^0L)^2L^{-3}k}$

20. $\dfrac{s^2ym^{-3}}{(s^0t^2)^{-3}m^{-3}st}$

21. $\dfrac{(x^{-3}yz^{-3})^2xy^0}{(xy^0z^{-2})^{-3}xy}$

22. $\dfrac{x^{-3}y^2xy^4}{(x^{-2}y)^3y^{-3}x}$

Simplify:

23. -3^{-2}

24. $\dfrac{1}{-2^{-3}}$

25. $-3^2 - [-2^0 - (3 - 2) - 2]$

26. $-2\{[-3 - 2(-2)][-2 - 3(-2)]\}$

27. $2\{-3^0[(-5 - 2)(-3) - 2]\}$

28. $-3[4^0 - 7(2 - 3) - 2^2]$

29. $-|-2 - 3| - (-5) - 3^3$

30. $-|-3^2 - 2| - 2^0 - (-3)$